1 Effect of Cycles on the Seed Path Probability

First, we consider the initial path from state 0 to state 1, where state 1 is the target state as in figure 1. Let's denote the probability of this path as P_{01}. Now, suppose a cycle occurs at state 0, where the path goes from state 0 to state 2 and then back to state 0. How would this cycle affect the probability of reaching state 1, or what would be the overall impact on the probability of reaching the target state?

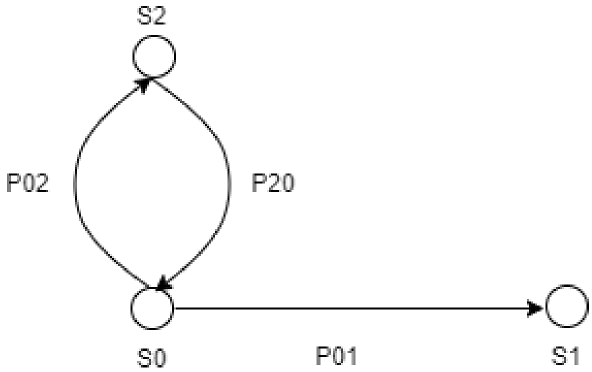

Figure 1: State Diagram For Scenario-1

We can consider several scenarios. First, there may be no cycles. Second, there could be a single cycle. Third, there might be two cycles. In fact, an infinite number of cycles could occur, assuming that eventually the cycle would break and the path would reach the target state. Let's denote P_n is the probability of the path with n cycles.

For $n = 0$:

$$P_0 = P_{01} \quad \text{(no cycle)}$$

For $n = 1$:

$$P_1 = P_{01}P_{02}P_{20}$$

For $n = 2$:

$$P_2 = P_{01}P_{02}P_{20}P_{02}P_{20} = P_{01}(P_{02}P_{20})^2$$

For $n = 3$:

$$P_3 = P_{01}(P_{02}P_{20})^3$$

$$\vdots$$

The total probability will be the sum of all path probabilities which can be given by the geometric series:

$$P_{\text{total}} = P_0 + P_1 + P_2 + P_3 = P_{01} + P_{01}(P_{02}P_{20}) + P_{01}(P_{02}P_{20})^2 + P_{01}(P_{02}P_{20})^3 + \cdots$$

The sum S of a geometric series

$$a + ar + ar^2 + ar^3 + \cdots \text{ is given by, } S = \frac{a}{1-r}.$$

Where a is the first term, and r is the common ratio.

So, we can write the total probability as below:

$$P_{\text{total}} = P_{01}\left(\frac{1}{1 - P_{02}P_{20}}\right)$$

It is to be noted that the series converges to a sum if and only if $|P_{02} \times P_{20}| < 1$. Otherwise, if $P_{02}P_{20}$ is equal to or greater than 1, the probability of escaping out the loop and reaching state 1 is zero, or not well defined.

Now let's consider another scenario where a cycle occurs at state 1 and another cycle occurs at state 2 as in the figure 2 where the target state is 3.

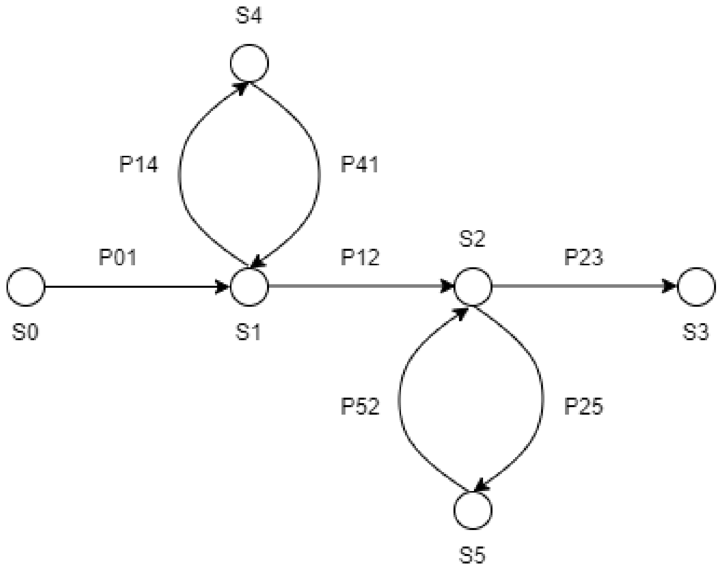

Figure 2: State Diagram For Scenario-2

In that case the the total path probability would be as below:

$$P_{\text{total}} = P_{01}P_{12}P_{23}\left(\frac{1}{1-P_{14}P_{41}}\right)\left(\frac{1}{1-P_{25}P_{52}}\right)$$

Now let's consider the figure 3 where we can generalize the scenarios where a be the probability up to a state where the cycle occurs and then build a sub-path up to one state beyond the cycle. Let that state be X and the probability be b for reaching X. Let R_1 and R_2 be the probabilities of the cycle.

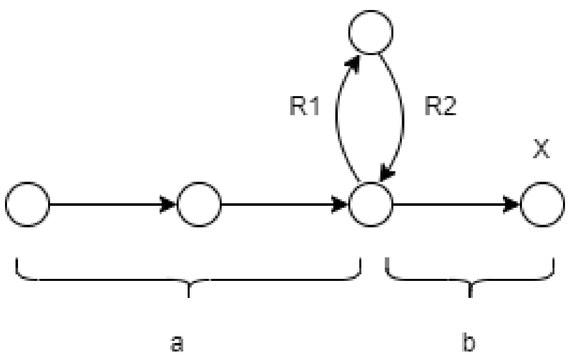

Figure 3: A generalized State Diagram

Then the probability of the sub-path up to x can be written as below:

$$P_{\text{total}} = ab\frac{1}{1-R_1R_2}$$

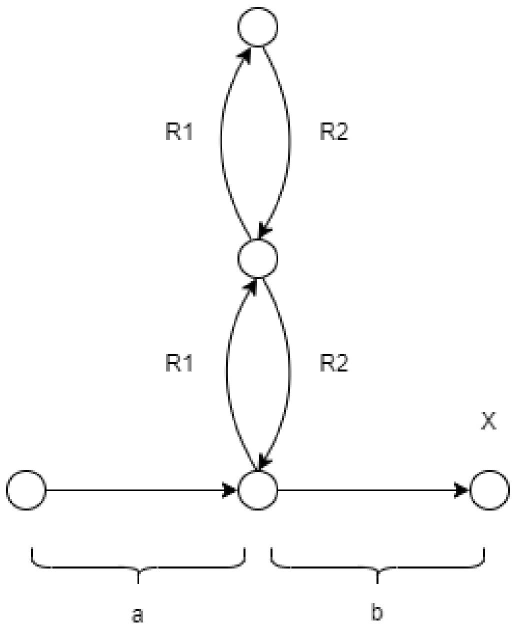

Figure 4: A generalized State Diagram With Recursive Cycles

Now, what if there is recursive cycles as in showed in figure 4? The seed path probability is ab and R_1R_2 is the cycle path probability. In this case there can be different possible paths to reach the target state. Let's consider the below two cases for now:

Case 1:

$$P_1 = aR_1R_2R_1R_2b = ab(R_1R_2)^2$$

Case 2:

$$P_2 = aR_1R_1R_2R_2b = ab(R_1R_2)^2$$

It appears that the effect remains the same regardless of whether the cycles recur or occur multiple times. Ultimately, the probability may result in the sum of a geometric series. Then the total probability would again look like below:

$$P_{\text{total}} = ab\frac{1}{1 - R_1R_2}$$

In conclusion, we can say that, all these scenarios we have considered so far have aggregated the total path probability of reaching the target state. In other words, cycles have increased the seed path probability in all these cases.